Lecture 8
Probability

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Random Experiments

- An experiment which may give different results, when they are repeated under same conditions.
- Examples: Tossing a number of coins
- Throwing number of dice
- Match between two teams
- Selection of team
- Selection of any item
- Selection of playing card from a deck (52 cards), one card, two cards, etc. With replacement and without replacement, Simultaneously selection, selection in turn.

Outcomes:

- Any possible result of a random experiment.
- Sample Space (S): The set of all possible outcomes of a random experiment:
- Tossing a coin:
- Tossing two coins:
- Tossing three coins:
- Tossing coins:

Examples

- Throwing one die:
- Throwing 2 dice:

- If we throw dice then .

Examples:

- Consider a deck of 52 cards. Two colors (Red-26, Black-26)
- Types 4 (Dimonds-13 J Q K, Hearts-13 J Q K, Spades-13 J Q K, Club-13 J Q K.

- Drawing one card.
- Drawing two cards simul…
- Drawing three cards…
- Drawing number of cards simul…

Examples:

- Drawing two cards in turn without replacement:
- Drawing three cards in turn without replacement:
- Drawing cards in turn without replacement:
- .
- Drawing two cards in turn with replacement:
- Drawing three cards in turn with replacement:
- Drawing cards in turn with replacement:

Event:

- Any subset of a sample space:
- Total happenings -- Sample space, Favorable happenings Event

- Tossing a coin:
- Samples space is known sure event.
- is known as impossible event

Mutually Exclusive Events:

- Two events and are said to ME if

- Examples:

Equally Likely:

- Two events and will be equally likely, if they have same chance of occurrence.

- Examples:

Exhaustive Events:

- Two ME events and will be EE if

- Examples:

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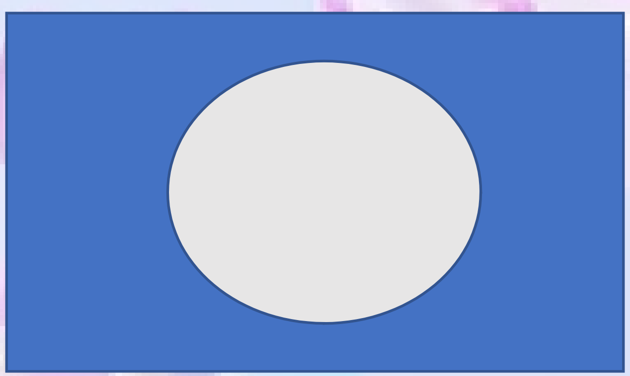
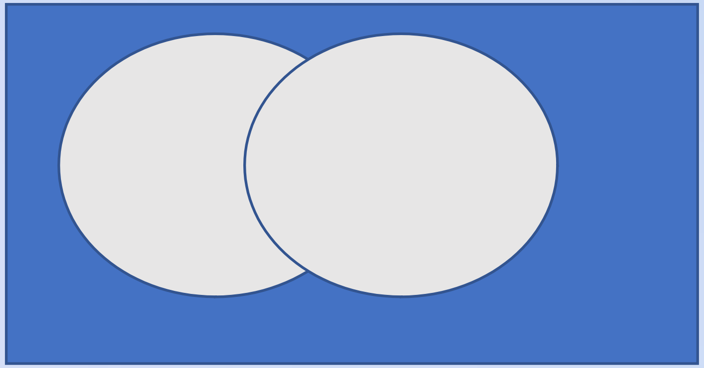
Venn’s Diagram

- Blue color (Rectangle) → Sample space
- White color (Circle) → Event





Definitions of Probability:

- **Classical Definition**:

For an event probability of an event is denoted by and is defined as:

- Probability of an event:

If and

- **The Relative Frequency Definition**:

If a random experiment is performed time (large) and an event is observed times, then

Experiments	Number of times coin is tossed	Number of Heads appeared	Ratio
Buffon			
K. Pearson			
K. Pearson			

Axiomatic Definition of Probability:

For a given event   will be probability of if

1) for any event
2) and
3) If then .

Subjective or Personalistic Definition:

- Subjective probability refers to the probability of something happening based on an individual's own experience or personal judgment.
- A subjective probability is not based on market data or historical information and differs from person to person.
- In other words, it is created from the opinion of an individual and is not based on fact.

Example:

- An individual is asked the probability of a dice roll yielding a 6. The individual looks at the past three rolls and notes that 6 came up in all instances. The individual believes that the probability of the next dice roll yielding a 6 is at 30%. Although the mathematical prediction is incorrect (the probability is 16.67%), the individual’s personal experience of the dice roll yielding 6 in three instances created a situation where he used subjective probability.

Example:

If a card is drawn from a deck of 52 playing cards, find the probability that the card is (i) red card (ii) the card is a diamond (iii) the card is a 10

Solution: Here

(i) Let be the event representing a red card then

(ii) Let be the event representing a diamond card then

(iii) Let be the event representing a 10, then

Example:

Two dice are thrown. What is the probability of getting (i) a double six (ii) a sum of 8 or more dots?

Solution: The sample space in this case will be:

(i) Let be the event having double six then

(ii) Let be the event that a sum is 8 or more dots, then

Example:

Six white balls and four black are placed in bag. If six ball are taken from the bag, what is the probability of their being three black balls?

Solution:

Example:

An employer wishes to hire three people from a group of 15 applicants, 8 men and 7 women. If he selects 3 person at random. What is the probability that (i) all are men (ii) at least on will be woman?

Solution: Here

(i) Let represent the event that contain 3 men then

(ii) Let be the event representing at least one woman then

Example:

Four items are taken at random from a box of 12 items and is inspected. The box is rejected if more than one item is found to be faulty. If there are three faulty items in the box, find the probability that the box is accepted.

Solution: Let be the sample space then

To find probability of accepting the box is equal to find probability of at the most one faulty item. Let be the event then

Laws of Probability

1- for

Proof: Since .

2- Law of Complementation: If and is the complement of then

Proof: Since and so using third axiom of probability definition we have

Conti…

- If a coin is tossed times and probability of head in a single trial is Then prob of at least one tail will be

Example 6.11: A coin is biased so that the probability that it falls showing tails is

(a) What is the probability of getting at least one head, when a coin is tossed 5 times.

(b) How many times the coin must be tossed so that the probability of getting at least one head is greater than 0.98?

Solution: Here

(c) Let the coin be tossed times and

So the coin should be tossed at least 14 times.

Cont.

3. If then

Proof: Note that and are mutually exclusive events and

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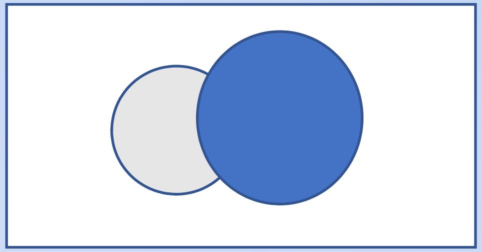


Addition Law

4. If

Proof: Since

Dividing by

If and are mutually exclusive, then

Example 6.12

If a card is drawn from a deck of 52 card. What is the probability that the card drawn is a club card or face card?

Solution: Let and be the events representing club cards and face cards respectively. Since

We need to find , which is given by

Example:

An integer is chosen at random from the first 200 positive integers. What is the probability that the integer chosen is divisible by 6 or 8?

Solution: Let be the sample space then

Let be the event that an integer chosen is divisible by 6, then

Let be the event that an integer chosen is divisible by 8, then

 will represent those integers, which are divisible by both 6 and 8, so

Example

Two natural events can result in the failure of a dam in an earthquake-prone area. Firstly, a very high flood, exceeding the design capability of its spillway, say, event A, may destroy it. Secondly, a destructive earthquake can cause a structural collapse, say, event B. Hydrological and seismological consultants estimate that the probability measures characterizing flood exceedance and earthquake occurrence on a yearly basis are , respectively. The occurrence of one or both events can result in the failure of the dam. What is the probability of failure of the dam?

Solution

Using formula:

Only the first two probabilities on the right are known. However, the engineer can assume that the joint event has an extremely low probability, so

If one takes, for example, values of a and b for small dams in seismic areas as 0.02 and 0.01, respectively,

This indicates a probability of not more than 3%

Example:

Three horses and are in a race. is twice as likely to win as and is twice as likely to win as . What is the probability that or wins?

Solution: Let , then ,

It must be noted that all the events and are mutually exhaustive so

Now

Theorem 6.6

- This can be proved by taking and applying addition law.
- See theorem 6.6 in Sher Muhammad Book.
- This is the probability of at least one of the events happening

Example

A card is drawn from a deck of 52 cards, what is the probability that the card drawn is a Heart, Face or King card?

Solution: Let and be the events representing Hearts, Faces and Kings cards.

Required:

Conditional Probability:

- Example 6.17. Two coins are tossed. What is the conditional probability that two heads result, given that there is at least one head ?
- Solution:

Example 6.19 What is the probability that a randomly selected poker hand, contains exactly 3 aces given that it contains *at least* 2 aces?

Let A represent the event that exactly 3 aces are selected and B, the event that *at least* 2 aces are selected. Then we need $P(A/B)$.

Since a poker hand consists of 5 cards, therefore the sample space S contains $\binom{52}{5} = 2{,}598{,}960$ outcomes.

$$n(A) = \binom{4}{3}\binom{48}{2} \text{ outcomes;}$$

$$n(B) = \binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}$$

as *at least* 2 aces means 2 or 3 or 4 aces; and

$$n(A \cap B) = \binom{4}{3}\binom{48}{2} \text{ as } A \subset B.$$

Thus $P(A \cap B) = \dfrac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}$, and

$$P(B) = \frac{\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}{\binom{52}{5}},$$

Hence $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$$= \frac{\binom{4}{3}\binom{48}{2}}{\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}},$$

$$= \frac{4{,}512}{108{,}336} = 0.0416.$$

Theorem 6.7 Multiplication Law. If A and B are any two events defined in a sample space, then

$$P(A \cap B) = P(A)\, P(B/A), \text{ provided } P(A) \neq 0,$$

$$= P(B)\, P(A/B), \text{ provided } P(B) \neq 0,$$

lary. This rule may be extended to several events. In case of three events A, B and C, we have

$$P(A \cap B \cap C) = P(D \cap C), \text{ were } D = A \cap B$$
$$= P(D)\, P(C/D)$$
$$= P(A \cap B)\, P(C/A \cap D)$$
$$= P(A)\, P(B/A)\, P(C/A \cap B);$$

Similarly, for more than three events, the formula may be proved by mathematical induction.

Example 6.20 A box contains 15 items, 4 of which are defective and 11 are good. Two items are ...d. What is the probability that the *first* is good and the *second* defective.

Let A represent the event that the first item selected is good and B, the event that the second item is ...ctive.

Then we need to calculate the probability of the joint event $A \cap B$ by the rule $...B) = P(A).\, P(B/A)$.

Now $P(A) = \frac{11}{15}$

Given the event A has occurred, there remains 14 items of which 4 are defective. Therefore the ...ity of selecting a defective after a good has been selected, *i.e.* $P(B/A) = \frac{4}{14}$.

Hence $P(A \cap B) = P(A).\, P(B/A) = \frac{11}{15} \times \frac{4}{14} = \frac{44}{210} = 0.16.$

Example: 6.21

- Two cards are dealt from a deck. What is the probability that second card is heart:

Solution: We have two possibilities in this case… 1) First card is heart and second is heart 2) First is not heart and second heart.

Example: 6.22

- Box A contains 5 green and 7 red balls. Box B contains 3 green, 3 red and 6 yellow balls. A box is selected at random, and a ball is drawn at random from it. What is the probability that the ball drawn is green?
- Solution:

There are two possibilities : A is selected and green ball is drawn or B is selected and green ball is drawn:

Example 6.23

An urn contains 10 white and 3 black balls. Another urn contains 3 white and 5 black balls. Two balls are transferred from first urn and placed in the second and then one ball is drawn from the later. What is the probability that it is a white ball?

- Two ball are drawn from A and placed in B. Possibilities are:

1) A: 2 black balls are transferred 2) B: 1 black and 1 white

3) C: 2 white

Example 6.25 Three urns of the same appearance are given as follows:

Urn A contains 5 red and 7 white balls.

Urn B contains 4 red and 3 white balls.

Urn C contains 3 red and 4 white balls.

An urn is selected at random and a ball is drawn from the urn.

i) What is the probability that the ball drawn is red?

ii) If the ball drawn is red, what is probability that it came from urn A?

i) Now the probability of drawing a red ball is given by the relation

$$P(R) = P(A)\,P(R/A) + P(B)\,P(R/B) + P(C)\,P(R/C)$$

as there are three mutually exclusive paths leading to the drawing of a red ball.

Hence $P(R) = \frac{1}{3}.\frac{5}{12} + \frac{1}{3}.\frac{4}{7} + \frac{1}{3}.\frac{3}{7}$

$$= \frac{119}{252} = 0.4722$$

ii) Here we need the probability that urn A is selected, given that the ball drawn is red, $P(A/R)$.

By definition, $P(A/R) = \dfrac{P(A \cap R)}{P(R)}$

But $P(A \cap R)$ = Probability that urn A is selected and a red ball is drawn

$$= \frac{1}{3} \times \frac{5}{12} = \frac{5}{36}$$

Hence $P(A/R) = \dfrac{5/36}{119/252}$

$$= \frac{35}{119} = 0.294$$

INDEPENDENT AND DEPENDENT EVENTS

Two events A and B in the same sample space S, are defined to be *independent* (*or statistically independent*) if the probability that one event occurs, is not affected by whether the other event has or has not occurred, that is

$$P(A/B) = P(A) \quad \text{and} \quad P(B/A) = P(B).$$

It then follows that two events A and B are independent *if and only if*

$$P(A \cap B) = P(A)\ P(B)$$

Example 6.26 Two events A and B are such that $P(A) = \frac{1}{4}, P(A/B) = \frac{1}{2}$, and $P(B/A) = \frac{2}{3}$.

i) Are A and B independent events?

ii) Are A and B mutually exclusive events?

iii) Find $P(A \cap B)$ and $P(B)$.

i) If A and B are independent events, then $P(A/B)=P(A)$

Now $P(A)=\dfrac{1}{4}$ and $P(A/B)=\dfrac{1}{2}$ i.e. $P(A/B)\neq P(A)$

Hence A and B are not independent events.

ii) If A and B are mutually exclusive events, then $P(A/B)=0$.

But it is given that $P(A/B)=\dfrac{1}{2}$

Hence A and B are not mutually exclusive events.

iii) Now $P(A\cap B)=P(A)\ P(B/A)$

$$=\frac{1}{4}\times\frac{2}{3}=\frac{1}{6}$$

By definition, we have

$$P(B)\ P(A/B)=P(A)\ P(B/A)$$

or $\quad P(B)\left(\dfrac{1}{2}\right)=\left(\dfrac{1}{4}\right)\left(\dfrac{2}{3}\right)$ so that $P(B)=\dfrac{1}{4}\times\dfrac{2}{3}\times\dfrac{2}{1}=\dfrac{1}{3}$

- See Examples 6.27 and 6.28 from Sher M C book.

Theorem 6.9 If A and B are two independent events in a sample space S, then (i) A and $\overline{B}$ are independent, (ii) $\overline{A}$ and B are independent, and (iii) $\overline{A}$ and $\overline{B}$ are independent.

Proof. (i) The events $A \cap B$ and $A \cap \overline{B}$ are mutually exclusive and their union is A
$A = (A \cap B) \cup (A \cap \overline{B})$.

Therefore $P(A) = P(A \cap B) + P(A \cap \overline{B})$

or $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

$= P(A) - P(A)P(B)$ [$\because$ A and B are independent]

$= P(A)\,[1 - P(B)] = P(A)\,P(\overline{B})$

Hence A are $\overline{B}$ are independent.

(ii) Similarly,

$P(B) = P(B \cap A) + P(B \cap \overline{A})$

or $P(B \cap \overline{A}) = P(B) - P(B \cap A)$

$= P(B) - P(B)\,P(A)$ [$\because$ A and B are independent]

$= P(B)\,[1 - P(A)] = P(B)\,P(\overline{A})$

Therefore $\overline{A}$ and B are independent.

(iii) Using De Morgan's law, $\overline{A} \cap \overline{B} = \overline{A \cup B}$, we have

$$
\begin{aligned}
P(\overline{A} \cap \overline{B}) &= P(\overline{A \cup B}) \\
&= 1 - P(A \cup B) \\
&= 1 - P(A) - P(B) + P(A \cap B) \\
&= 1 - P(A) - P(B) + P(A)\,P(B) \\
&= [1 - P(A)][1 - P(B)] = P(\overline{A})\,P(\overline{B})
\end{aligned}
$$

which shows that $\overline{A}$ and $\overline{B}$ are independent.

Example 6.29 Two cards are drawn from a well-shuffled ordinary deck of 52 cards. Find the probability that they are both aces if the first card is (i) replaced, (ii) not replaced.

(P.U., B.A./B.Sc., 1967; M.A. Econ. 1969)

Let A denote the event *ace* on first draw and B denote the event *ace* on the second draw.

(i) In case of replacement, event A and B are independent.

Thus $P(\text{both cards are aces}) = P(A \cap B) = P(A)\,P(B)$

$$= \frac{4}{52} \times \frac{4}{52} = \frac{1}{169}.$$

(ii) If the first card is not replaced, then A and B are dependent events and therefore

$P(\text{both cards are aces}) = P(\text{first card is an ace}) \times P(\text{second card is an ace given that the first card is an ace})$

i.e. $P(A \cap B) = P(A)\,P(B/A) = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}.$

Example 6.31 The probability that a man will be alive in 25 years is 3/5, and the probability that his wife will be alive in 25 years is 2/3. Find the probability that (i) both will be alive, (ii) only the man will be alive, (iii) only the wife will be alive, (iv) at least one will be alive and (v) neither will be alive in 25 years.

(P.U., B.A./B.Sc. 197

Let A be the event that the man will be alive and B be the event that his wife will be alive in 25 years. Then

$$P(A) = \frac{3}{5}, \text{ and } P(B) = \frac{2}{3}.$$

i) We need the probability that both will be alive, *i.e.* $P(A \cap B)$.

Since A and B are independent, therefore

$$P(A \cap B) = P(A).\ P(B) = \frac{3}{5} \times \frac{2}{3} = \frac{2}{5}.$$

ii) We need the probability that only the man will be alive, *i.e.* $P(A \cap \overline{B})$.

Since A and $\overline{B}$ are independent and $P(\overline{B}) = 1 - P(B)$, therefore

$$P(A \cap \overline{B}) = P(A)\ P(\overline{B}) = \frac{3}{5} \times \left(1 - \frac{2}{3}\right) = \frac{1}{5}.$$

iii) We require the probability that only the wife will be alive, i.e. $P(\overline{A} \cap B)$. Thus

$P(\overline{A} \cap B) = P(\overline{A})\ P(B) = \frac{2}{5} \times \frac{2}{3} = \frac{4}{15}$, as the event $\overline{A}$ and B are independent

$P(\overline{A}) = 1 - P(A)$.

iv) We require the probability that *at least* one will be alive, i.e. $P(A \cup B)$.

Since the events A and B are independent and not mutually exclusive, therefore

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{3}{5} + \frac{2}{3} - \frac{2}{5} = \frac{13}{15}.$$

v) We need the probability that neither will be alive, i.e. $P(\overline{A} \cap \overline{B})$.

Since $\overline{A}$ and $\overline{B}$ are independent, therefore

$$P(\overline{A} \cap \overline{B}) = P(\overline{A})\,P(\overline{B})$$

$$= [1 - P(A)]\,[1 - P(B)]$$

$$= \frac{2}{5} \times \frac{1}{3} = \frac{2}{15}.$$

Theorem 6.11 Bayes' theorem. If the events $A_1, A_2, \ldots, A_k$ form a partition of sample space S that is, the events A_i are mutually exclusive and their union is S, and if B is any other event of S such that it can occur only if one of the A_i occurs, then for any i,

$$\boxed{P(A_i/B) = \frac{P(A_i)P(B/A_i)}{\sum_{i=1}^{k} P(A_i)P(B/A_i)}, \text{ for } i = 1, 2, \ldots, k.}$$

Proof: By the multiplicative law of probabilities, we have

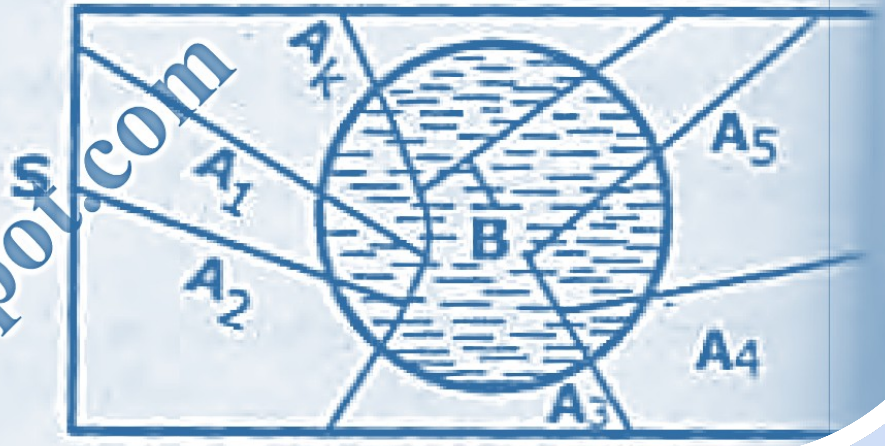
$$P(B \cap A_i) = P(B)P(A_i/B)$$

$$= P(A_i)P(B/A_i).$$

Equating the equivalent relations of $P(B \cap A_i)$ and solving for $P(A_i/B)$, we get

$$P(A_i/B) = \frac{P(A_i)P(B/A_i)}{P(B)}$$

We may write the event B as $B = S \cap B$ (see the Venn diagram)



$$= (A_1 \cup A_2 \cup \ldots \cup A_k) \cap B$$

$$= (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_k \cap B),$$

where the $A_i \cap B$ are also mutually exclusive.

B is shaded

Therefore $P(B) = P(A_1 \cap B) + P(A_2 \cap B) + ... + P(A_k \cap B)$

Using the multiplicative law of probabilities, we may express each term $P(A_i \cap B)$ $P(A_i)P(B/A_i)$. Then

$$P(B) = P(A_1)P(B/A_1) + P(A_2)P(B/A_2) + ... + P(A_k)P(B/A_k)$$

$$= \sum_{i=1}^{k} P(A_i)\, P(B/A_i)$$

This result is generally known as the *theorem on total probability*. Replacing $P(B)$ by the probability formula for the event B, we obtain *Bayes' formula* as

$$P(A_i/B) = \frac{P(A_i)P(B/A_i)}{\sum_{i=1}^{k} P(A_i)P(B/A_i)}$$

Example 6.34 In a bolt factory, machines A, B and C manufacture 25, 35 and 40 percent of the output, respectively. Of their outputs, 5, 4, 2 percent, respectively, are defective bolts. A bolt is [illegible] at random and found to be defective. What is the probability that the bolt came from machine A?

The *a priori* probabilities (before the information that the bolt is defective) are $= 0.25$, $P(B) = 0.35$, and $P(C) = 0.40$.

Let E represent the event that a bolt is defective (D).

Then the conditional probabilities are

$$P(E/A) = 0.05,\ P(E/B) = 0.04, \text{ and } P(E/C) = 0.02.$$

The outcomes with their respective probabilities may be shown by a tree diagram as below:

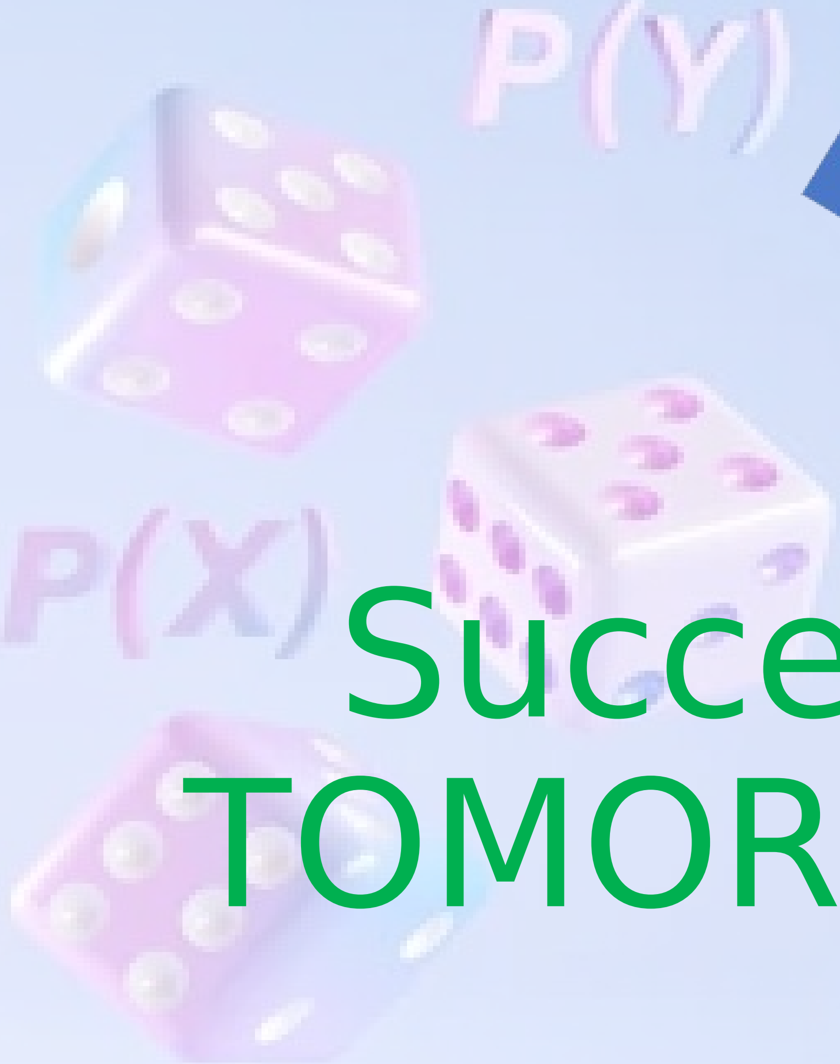
E) is the *a posteriori* probability that the selected defective bolt came from machine A.

by Bayes' theorem, we get

$$E) = \frac{P(A).P(E/A)}{P(A).P(E/A) + P(B).P(E/B) + P(C).P(E/C)}$$

$$= \frac{(0.25)(0.05)}{(0.25)(0.05) + (0.35)(0.04) + (0.40)(0.02)}$$

$$= \frac{0.0125}{0.0345} = 0.362$$



Success will not come TOMORROW unless you start TODAY...